MATH 20900 Final Review

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1 Banach-Tarski, Baire Category Theorem

Theorem 1.1 *C is a Cantor set if and only if it is compact, has empty interior, and has no isolated points.*

Theorem 1.2 (Banach-Tarski Theorem) *Given $A, B \subseteq \mathbb{R}^n$ (with $n \geq 3$), bounded and with nonempty interior, we can write $A = A_1 \cup ... \cup A_m$ and $B = B_1 \cup ... \cup B_m$ so that $i \neq j \implies A_i \cap A_j = \emptyset = B_i \cap B_j$ and each A_i is isometric to B_i.*

Corollary 1.2.1 *There is no finitely-additive measure defined on all subsets of $\mathbb{R}^3$ that is invariant under isometry and nonzero on the unit ball.*

Definition 1.1 (Categories) *Let X be a topological space, and consider $A \subseteq X$. We say that*

- *A is nowhere dense if $\overline{A}$ has empty interior.*
- *A is of first category if $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense.*
- *A is of second category if it is not of first category.*
- *A is residual if A^c is of first category.*

Proposition 1.1 (Basic properties of categories) *Let X be a topological space, and consider $A \subseteq X$.*

- *If A is first category and $B \subseteq A$, then B is also first category.*
- *If A_n is first category and $A = \bigcup A_n$, then A is also first category.*
- *If A is residual and $A \subseteq B$, then B is also residual.*

- *If A_n is residual and $A = \bigcap A_n$, then A is also residual.*

Definition 1.2 (Baire space) *We say that X is a Baire space if every open nonempty set is of second category.*

Theorem 1.3 (Baire Category Theorem) *If X is a complete metric space, then X is a Baire space.*

Theorem 1.4 *For any Borel set $A \subseteq \mathbb{R}$, there exists an interval I such that either A or A^c is first category in I.*

Definition 1.3 (Baire set) *We say that a set A is Baire if there exists an open set G such that $(A \setminus G) \cup (G \setminus A)$ is of first category.*

Theorem 1.5 *If A is a Borel set, then A is Baire.*

Let $C([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$ with the norm $||f|| = \sup |f(x)|$. We know that this is a complete metric space. Recall that we say a property is true of a 'typical continuous function' if the set of functions with that property is residual in $C([0,1])$.

Theorem 1.6 *A typical continuous function is not monotone on any interval.*

Definition 1.4 (Locally monotone) *We say that a function f is locally monotone (increasing) at x if there exists $\varepsilon > 0$ such that for any $y \in (x - \varepsilon, x + \varepsilon)$, $y > x \implies f(y) \geq f(x)$ and $y < x \implies f(y) \leq f(x)$.*

Theorem 1.7 *A typical continuous function is not locally monotone at any point $x \in [0,1]$.*

Corollary 1.7.1 *A typical continuous function is nowhere differentiable.*

Proposition 1.2 *The following are all typical properties of a continuous function f:*

- *no level sets of f contain intervals*
- *each level set has at most one isolated point*
- *for all but countably many c, $\{x | f(x) = c\}$ is a Cantor set*
- *there exist countably many c such that the level set is a Cantor set and an isolated point*

2 Basic Measure Theory

Definition 2.1 (Measure) *Let X be a set. A measure on X is a function $\mu : \mathcal{P}(X) \to \mathbb{R}$, where $A \mapsto \mu(A)$. It satisfies the following properties:*

- *Countable additivity - if A_n are pairwise disjoint for $n \in \mathbb{N}$ then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

- *Nonnegativity - $\mu(A) \geq 0$. (Otherwise it is called a signed measure.)*
- $\mu(\emptyset) = 0$

Typically, however, μ is defined only on a $\sigma-$algebra in X.

Definition 2.2 (σ-algebra) *A σ-algebra on X is a collection $\Sigma \subseteq \mathcal{P}(X)$ that satisfies the following properties:*

- $\emptyset \in \Sigma$
- $A \in \Sigma \implies A^c \in \Sigma$
- $A_n \in \Sigma \implies \bigcup_{n=1}^{\infty} A_n \in \Sigma$

Definition 2.3 (Borel sets) *If X is a topological space, the Borel sets are the σ-algebra generated by the open sets. Moreover, we call countable intersections of open sets G_δ and countable unions of closed sets F_σ.*

Definition 2.4 (Lebesgue measure) *To construct Lebesgue measure, we begin with the σ-algebra Σ_0 generated by finite unions of boxes in $\mathbb{R}^n$. If $I_k = [a_k, b_k]$ are intervals, we define an outer measure by $\mu_0(I_1 \times ... \times I_n) = \prod_{i=1}^{n}(b_i - a_i)$. Lebesgue measure ($\mu$) and its corresponding σ-algebra Σ are thus generated by the Caratheodery extension algorithm. Recall that we defined*

$$\mu(A) = \inf\{\sum_{n=1}^{\infty} \mu_0(A_n) | A_n \in \Sigma_0, A \subseteq \bigcup_{n=1}^{\infty} A_n\}$$

$$\Sigma = \{A \subseteq X | \forall S \subseteq X, \mu(A \cap S) + \mu(S \setminus A) = \mu(S)\}$$

Theorem 2.1 *If $A_1 \subseteq A_2 \subseteq ...$, then $\mu(\bigcup A_n) = \lim_{n\to\infty} \mu(A_n)$.*

Theorem 2.2 *Suppose $A_1 \supseteq A_2 \supseteq \ldots$ and there exists $k \in \mathbb{N}$ such that $\mu(A_k) < \infty$, then $\mu(\bigcap A_n) = \lim_{n\to\infty} \mu(A_n)$.*

Proposition 2.1 *For any $A \in \Sigma$ and $\varepsilon > 0$, there exists an open set $G \subseteq \mathbb{R}^n$ such that $\mu(G \setminus A) < \varepsilon$.*

Definition 2.5 (Radon measure) *Let (X, μ) be a metric space with an outer measure. We call it*

- *locally finite if for any $x \in X$, there exists $r > 0$ such that $\mu(B(x, r)) < \infty$.*
- *Borel if Borel sets are measurable.*
- *Borel regular if it is Borel and for any $A \subseteq X$, there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.*
- *inner regular if we can write*

  $$\mu(G) = \sup_{K \subseteq G} \mu(K)$$

  where G is open and K is compact.
- *outer regular if we can write*

  $$\mu(A) = \inf_{A \subseteq G} \mu(G)$$

  where G is open.
- *Radon if it is Borel, inner regular, outer regular, and $\mu(K) < \infty$ for any compact set K.*

Proposition 2.2 *Lebesgue measure satisfies all of these, and in particular, is a Radon measure.*

Proposition 2.3 *Given a separable complete metric space X, if an outer measure is locally finite and Borel regular, then it is Radon.*

Definition 2.6 *Consider a function $f : X \to \mathbb{R}$, a σ-algebra Σ on X, and a measure on X. We call the function*

- *measurable if $f^{-1}(G) \in \Sigma$ for any open $G \subseteq \mathbb{R}$.*
- *μ-measurable if $f^{-1}(G)$ is measurable for any open $G \subseteq \mathbb{R}$.*

- *Borel if $f^{-1}(G)$ is Borel for any open $G \subseteq \mathbb{R}$.*
- *Baire if $f^{-1}(G)$ is Baire for any open $G \subseteq \mathbb{R}$.*

Proposition 2.4 *There exists a non-measurable set in $\mathbb{R}$.*

Proposition 2.5 *There exist measurable functions f, g such that $f \circ g$ is not measurable.*

Theorem 2.3 (Egorov's Theorem) *Suppose $f_n \to f$ almost everywhere and f_n, f are supported on E, where $\mu(E) < \infty$. Then for any $\varepsilon > 0$, there exists a closed set $B_\varepsilon \subseteq E$ such that $f_n \rightrightarrows f$ uniformly on B_ε and $\mu(E \setminus B_\varepsilon) < \varepsilon$.*

Theorem 2.4 (Luzin's Theorem) *Let f be a measurable function defined on E, where $\mu(E) < \infty$. Then for any $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subseteq E$ such that $\mu(E \setminus F_\varepsilon) < \varepsilon$ and f restricted to F_ε is continuous.*

Proposition 2.6 *If $A \subseteq R^n$ is Lebesgue-measurable, then there exists a G_δ set H such that $A \subseteq H$ and $\mu(H \setminus A) = 0$.*

Proposition 2.7 *If $A \subseteq R^n$ is Lebesgue-measurable, then for any $\varepsilon > 0$, there exists a closed set $F \subseteq A$ such that $\mu(A \setminus F) < \varepsilon$. Moreover, there exists an F_σ set $K \subseteq A$ such that $\mu(A \setminus K) = 0$.*

3 Integration and Convergence Theorems

Theorem 3.1 *A function $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its discontinuity set is a null set. Moreover, if f is Riemann integrable, then it is measurable.*

Definition 3.1 (Lebesgue integration) *First, let us consider a simple function f, i.e.*

$$f = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where $c_i \in \mathbb{R}$ and the E_i are measurable. Then we define

$$\int f \, d\mu = \sum_{i=1}^{n} c_i \mu(E_i)$$

Second, if $f \geq 0$ and is measurable, then we define

$$\int f \, d\mu = \sup_{0 \leq g \leq f} \int g \, d\mu$$

where g is a simple function. Finally, if f is measurable, we can decompose it as $f = f^+ - f^-$, which are everywhere nonnegative, and define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

Definition 3.2 *We say that f is integrable if $\int |f| \, d\mu < \infty$.*

Theorem 3.2 (Linearity of integration) *If f, g are integrable and $\alpha, \beta \in \mathbb{R}$, then*

$$\int \alpha f + \beta g \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$$

Theorem 3.3 (Monotone Convergence Theorem) *Let f_n be measurable functions, where $0 \leq f_n(x) \leq f_{n+1}(x)$ almost everywhere, for all $n \in \mathbb{N}$. Then if $f_n \to f$ pointwise almost everywhere,*

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proposition 3.1 *If $f \geq 0$ is measurable, then there exist simple functions s_n such that $s_n \to f$ almost everywhere and $s_n \leq s_{n+1}$.*

Lemma 3.4 (Fatou's Lemma) *If $f_n \geq 0$ are measurable functions and $f = \liminf(f_n)$, then*

$$\lim_{n \to \infty} \int f_n \, d\mu \geq \int f \, d\mu$$

Theorem 3.5 (Beppo-Levi) *If $f_n \geq 0$ are measurable functions, then*

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu$$

Theorem 3.6 (Dominated Convergence Theorem) *Suppose f_n are measurable functions such that $f_n \to f$ (pointwise) almost everywhere and there exists an integrable function $G \geq 0$ such that $|f_n| \leq G$, for all n. Then*

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$$

Theorem 3.7 (Fubini's Theorem) *Consider measure spaces (X, μ) and (Y, ν) (where these are σ-finite) , and a function $f : X \times Y \to \mathbb{R}$. Moreover, suppose that*

$$\int_{X\times Y} |f(x, y)|\, d(\mu \times \nu)(x, y) < \infty$$

Then

$$\int_Y \int_X f(x, y)\, d\mu(x)d\nu(y) = \int_X \int_Y f(x, y)\, d\nu(y)d\mu(x) = \int_{X\times Y} f(x, y)\, d(\mu\times\nu)(x, y)$$

4 Hausdorff Dimension

Definition 4.1 (Hausdorff measure) *Given a set $E \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, we define*

$$H^s_\varepsilon(E) = \inf_{E\subseteq \bigcup E_n} \sum_{n=1}^{\infty} diam(E_n)^s$$

where $diam(E_n) < \varepsilon$ for all $n \in \mathbb{N}$. Then the s-dimensional Hausdorff measure of E is given by

$$H^s(E) = \lim_{\varepsilon\to 0} H^s_\varepsilon(E)$$

Theorem 4.1 (Hausdorff dimension) *For any $E \subseteq \mathbb{R}^n$, there exists a constant $0 \le s_0 \le n$ such that for any $t < s_0$, $H^t(E) = \infty$ and for any $t > s_0$, $H^t(E) = 0$. We call s_0 the Hausdorff dimension of E. Equivalent to this theorem is the combination of the following two statements:*

- *If $H^s(E) < \infty$ for some s, and $t > s$, then $H^t(E) = 0$.*
- *If $H^s(E) > 0$ for some s and $t < s$ then $H^t(E) = \infty$.*

Theorem 4.2 *The dimension of the middle-thirds Cantor set (C) is $s = \frac{\log(2)}{\log(3)}$. That is, $0 < H^s(C) < \infty$.*

5 Covering Theorems

Theorem 5.1 (Vitali Covering Theorem) *Consider a set $A \subseteq \mathbb{R}^n$, and a collection of balls such that for each $x \in A$, there exist arbitrarily small balls centered at x. Then there exists a disjoint collection of balls covering almost all of A.*

Theorem 5.2 (Besicovitch Covering Theorem) *Consider any collection of balls $B_\alpha \subseteq \mathbb{R}^n$. Suppose that either the radii of the balls are bounded or the centers form a bounded set. Then we can choose $c(n)$ (some constant depending only on the dimension) countable subclasses of disjoint balls B_{ij}, where $i \in [c]$ and $j \in \mathbb{N}$, so that $\bigcup B_{i,j}$ covers the centers of the balls.*

6 Density and Differentiation

Definition 6.1 (Lebesgue Density) *Given a set $A \subseteq \mathbb{R}^n$ and a point x, we define the density of x in A by*

$$d(A, x) = \lim_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))}$$

(Whenever this limit exists.)

Theorem 6.1 *If μ is a Radon measure, then for almost every $x \in A$, $d(A, x) = 1$. Moreover, for almost every $x \notin A$, $d(A^c, x) = 1$.*

Proposition 6.1 *A is measurable*

- *if and only if $d(A, x) = 0$ for almost every $x \in A^c$.*
- *if and only if $d(A^c, x) = 0$ for almost every $x \in A$.*

Definition 6.2 *If $A \subseteq H$ and μ is an outer measure such that $\mu(A) = \mu(H)$, then we say that H is a hull of A.*

Theorem 6.2 (Steinhaus Theorem) *Given $A \subseteq \mathbb{R}^n$ such that $\mu(A) > 0$, $A - A = \{x - y : x, y \in A\}$ contains some ball $B(0, \varepsilon)$.*

Definition 6.3 (Density Topology) *We say that $A \subseteq \mathbb{R}^n$ is d-open (i.e. open in the density topology) if it is Lebesgue-measurable and for all $x \in A$, $d(A, x) = 1$. (We showed in class that this definition satisfies the properties of a topology.)*

Definition 6.4 (Bounded Variation) *Given a function $f : [0, 1] \to \mathbb{R}$, we say that it is of bounded variation if there exists a constant $M > 0$ such that for any partition $0 = x_0, x_1, ..., x_n = 1$,*

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M$$

Definition 6.5 (Luzin) *We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is Luzin, or N, if the image of every null set is null.*

Definition 6.6 (Absolute Continuity) *Given $f : \mathbb{R} \to \mathbb{R}$, we call it absolutely continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any collection of pairwise disjoint intervals $[x_i, y_i]$,*

$$\sum y_i - x_i < \delta \implies \sum |f(y_i) - f(x_i)| < \varepsilon$$

Theorem 6.3 (Absolute Continuity of Lebesgue Integration) *If $f \in L^1(\mathbb{R}^n, \mu)$ then for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\mu(A) < \delta \implies \int_A |f| \, d\mu < \varepsilon$$

Theorem 6.4 *If $f : [0, 1] \to \mathbb{R}$ is a function, the following statements are all equivalent:*

- *f is absolutely continuous*
- *f is Luzin and of bounded variation*
- *f is differentiable almost everywhere and $\int_0^x f'(t) \, d\mu(t) = f(x)$*
- *there exists a measure μ such that $\mu << \lambda$ and $f(x) = \mu(-\infty, x)$*

Definition 6.7 (Hardy-Littlewood Maximal Operator) *Given a function f, we define $M : f \mapsto Mf$ by*

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu$$

Definition 6.8 (Differential Bases) *In the definition of the Hardy-Littlewood Maximal Operator, we can choose to vary over:*

- *$B(x, r)$, balls centered at $x \implies$ this is the symmetric basis.*
- *$B \ni x$, any ball containing at $x \implies$ this is the uncentered basis.*
- *axis parallel rectangles containing $x \implies$ this is the strong basis.*

Lemma 6.5 (Weak (1,1) Estimate) *For any $t > 0$,*

$$\mu(\{x : |Mf(x)| > t\}) \leq \frac{c}{t} ||f||_1$$

where c is a constant depending only on the dimension. (This holds for the symmetric and uncentered differential bases, but not the strong basis.)

Theorem 6.6 (Interpolation Theorem) *Suppose M is a maximal operator satisfying the following:*

- $||Mf||_\infty \lesssim ||f||_\infty$
- $M(f_1 + f_2) \leq Mf_1 + Mf_2$
- *the weak 1-1 estimate*

Then there exists a constant c (depending only on dimension) such that

$$||Mf||_p \leq c||f||_p$$

Theorem 6.7 (Strong (p,p) Estimate) *If $p > 1$, there exists a constant c (depending only on dimension) such that*

$$||Mf||_p \leq c||f||_p$$

(This holds for any differential basis.) [Note that this theorem is a nontrivial corollary of the Interpolation Theorem because it holds for the strong differential basis, even though it does not satisfy the weak 1-1 estimate.]

Corollary 6.7.1 *For $p > 1$, all differential bases we have seen differentiate L^p functions.*

Theorem 6.8 (Lebesgue Differentiation Theorem) *Given $f \in L^1(\mathbb{R}^n)$ and μ is a Radon measure, we have that the following holds almost everywhere:*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(t)\, d\mu(t) = f(x)$$

Proposition 6.2 *In dimension at least 2, the strong differential basis does not differentiate L^1 functions, and the weak 1-1 estimate fails for the corresponding maximal operator.*

7 Decomposition Theorems

Consider a topological space X with two finite nonnegative Radon measures μ and ν.

Definition 7.1 (Absolutely Continuous Measures) *We say that ν is absolutely continuous w.r.t μ (and write $\nu << \mu$) if for any $A \in \Sigma$, $\mu(A) = 0 \implies \nu(A) = 0$.*

Definition 7.2 (Singular Measures) *We say that ν is singular w.r.t μ (and write $\nu \perp \mu$) if there exist disjoint sets A, B such that $X = A \cup B$ and $\mu(A) = 0 = \nu(B)$.*

Theorem 7.1 (Lebesgue Decomposition Theorem) *Given two such measures ν, μ, we can always uniquely decompose it as $\nu = \nu_a + \nu_s$ so that $\nu_a << \mu$ and $\nu_s \perp \mu$. Moreover, there exists a set $A \subseteq X$ such that $\nu_a = \nu|_{A^c}$ and $\nu_s = \nu|_A$.*

Theorem 7.2 (Jordan Decomposition) *Let μ be a finite (potentially signed) Radon measure. Then we can always write it as $\mu = \pi - \nu$, where $\pi, \nu \geq 0$. π is called the positive variation, ν is called the negative variation, and $\tau = \pi + \nu$ is called the total variation. Moreover, they are defined by*

$$\pi(A) = \sup_{S \subseteq A} \mu(S)$$

$$\nu(A) = - \inf_{S \subseteq A} \mu(S)$$

Theorem 7.3 (Hahn Decomposition Theorem) *Let μ be a signed measure, and let us write $\mu = \pi - \nu$. Then there exists sets P and N such that $X = P \cup N$, $P \cap N = \emptyset$, and $\pi(N) = 0 = \nu(P)$.*

Definition 7.3 (Absolute Continuity of Signed Measures) *If μ, ν are signed measures, we say that ν is absolutely continuous w.r.t μ if their total variations satisfies the original definition for nonnegative measures. That is, if $\mu|_A \equiv 0$, then $\nu|_A \equiv 0$.*

Definition 7.4 (Singularity for Signed Measures) *If ν, μ are signed measures, we say that ν is singular w.r.t μ if their total variations satisfy the original definition. That is, we can write $X = A \cup B$ where $A \cap B = \emptyset$ and $\mu|_A \equiv 0 \equiv \nu|_B$.*

Theorem 7.4 *If $\mu \perp \nu$ then*

$$\lim_{r \to \infty} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

is 0 $\mu-$almost everywhere and ∞ ν-almost everywhere.

Theorem 7.5 (Radon-Nikodym Theorem) *Suppose ν, μ are nonnegative finite measures. Then we can write*

$$\nu = \int f \, d\mu + \nu_s$$

where the integral takes the role of ν_a and $f \in L^1$ is called the Radon-Nikodym derivative.

8 L^p spaces

Definition 8.1 *Consider a function $f : X \to \mathbb{R}$, and a measure μ. For $1 \leq p < \infty$, we define*

$$||f||_p = (\int_X |f|^p \, d\mu)^{\frac{1}{p}}$$

For $p = \infty$, we define

$$||f||_\infty = \sup\{M : \mu(\{x : |f(x)| > M\}) = 0\}$$

Definition 8.2 *Given a finite measure space (X, μ), we say that $f \in L^p$ if $||f||_p < \infty$.*

Proposition 8.1 *If (X, μ) is a finite measure space and $p < q$, then $L^p \supseteq L^q$.*

Lemma 8.1 (Holder's Inequality) *If $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, and $g \in L^q$, then $fg \in L^1$ and $||fg||_1 \leq ||f||_p ||g||_q$.*

Lemma 8.2 (Minkowski's Inequality) *If $f, g \in L^p$, then $||f+g||p \leq ||f||_p + ||g||_p$.*

Proposition 8.2 *We have the following facts:*

- *L^∞ is dense in L^p.*
- *Simple functions are dense in L^∞ w.r.t the L^p norm.*
- *Continuous functions are dense in L^p.*

Theorem 8.3 (Riesz-Fischer Theorem) *L^p is a complete metric space, and thus a Banach space.*

Theorem 8.4 (Duality of L^p Spaces) *If $\frac{1}{p} + \frac{1}{q} = 1$, then $(L^p)^* \cong L^q$.*